The Adaptive Spectral Element Method and Comparisons with More Traditional Formulations for Ocean Modeling

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ABSTRACT

Triangular spectral elements offer high accuracy in complex geometries, but solving the related matrix problem can be cumbersome and time consuming. In restricted applications, recent developments have led to a family of discontinuous Galerkin formulations in which each element of the mesh leads to a local matrix problem. The main restriction is that all the fluid equations must be prognostic and solved explicitly in time. Such is the case for a hydrostatic ocean with a free surface and a Boussinesq approximation. Furthermore, there is a strong need for variable resolution in ocean modeling since the width of synoptic eddies and strong western currents such as the Gulf Stream are nearly two orders of magnitude smaller than the typical width of an ocean. Triangular elements also offer high flexibility for the adaptive problem. Some applications for shallow water test case problems are shown with comparisons to a traditional finite-difference model and to a finite-element coastal ocean model. These comparisons are made in rectangular domains where the finite-difference method has an inherent advantage. For a nonlinear wind-driven application, the spectral element model proved to be more expensive to run at reasonable accuracy than a second-order-accurate finite-difference model. Nonetheless the spectral element model appears to be a large improvement compared to finite-element models of low order, an encouraging result. A simple adaptive strategy is also investigated, with favorable results.

1. Introduction

The understanding of global oceanic currents and, ultimately, of climate variability requires the use of computer modeling. However, questions arise concerning the accuracy of such modeling, especially in the presence of complex geometrical configurations (straits, capes, etc.). Finite-difference (FD) models have been the basis of ocean modeling for the last 30 years (Bryan 1969). These models were and are very useful in getting relatively accurate solutions for a small amount of effort in coding and computation. Nonetheless, certain applications require irregular variable resolution or irregular coastline orientation that finite-difference models cannot accommodate easily (Adcroft and Marshall 1998). The size of resolved straits in finite-difference models, for instance, can only take values from a discrete set and requires tuning (Roberts and Wood 1997) in order to ensure an approximate representation. Moreover, these models do not offer local refinement capabilities for resolving straits, eddies, or strong unsteady currents like the Gulf Stream, although some nested strategies have been proposed (Blayo and Debreu 1999; Wadley and Bigg 1999). These features are barely resolved in today’s global ocean models.

The need for irregular resolution and geometry motivated the search for numerical methods that are defined on completely irregular geometries. In the last 30 years, developments in finite-volume (FV) and finite-element (FE) methods for industrial flows have been significant. It is of great interest to explore the capability of these numerical methods to model ocean physics and dynamics. The stability of these methods (viz., that associated with the existence of spurious pressure modes) is a complex issue. The stability is not particularly good if the variables are not staggered in space as is usually done in most FD models. The latter may use the so-called Arakawa B or C grids (Batteen and Han 1981). The FE stability issue was analyzed by Ladyzhenskaya (1969), Babuska (1971), and Brezzi (1974). They came up with a criterion known as the LBB condition valid for Stokes flows. Essentially, in order to satisfy the LBB condition the maximum polynomial order for approximating pressure (or elevation in the shallow water context) has to be lowered, compared to velocity (Fortin and Fortin 1985; Idelsohn et al. 1995; Le Roux et al. 1998; Hanert et al. 2003). To some degree, this is similar to staggering

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variables in space, at least for low-order FE methods. Despite much attention on FE stability, nonstaggered FE schemes are still very popular for reasons related to the ease of programming, plotting solutions, or parameterizing the physical processes (Peraire et al. 1986; Lynch and Werner 1991). They require some sort of stabilization and these formulations are unfortunately dissipative, at least for the type of applications that we are focusing on in this paper. As far as FV methods are concerned, two recent formulations have been proposed that apply to the ocean (Casulli and Walters 2000; Chen et al. 2003). Both are based on a staggered mesh and should, therefore, be nondissipative.

In terms of cost, most FE methods are hampered by requiring the solution of a (sparse) matrix problem even when the equations are solved explicitly in time. Therefore, some prefer to “jump” the matrix that results from the explicit time integration (Lynch and Werner 1991). This method gives a diagonal matrix with some cost in accuracy and, in effect, is equivalent to reducing the quadrature rules used in computing integrals over the triangular elements. Ultimately though, from the point of view of efficiency, since existing oceanic FE or FV models are first- to second-order accurate, the cost of increasing the accuracy may be larger than for second-order FD models (this does not take into account the boundary representation). The efficiency problem can be amplified by the sensitivity of these low-order methods to the regularity of the mesh.

It is thus challenging to have irregular geometry and variable resolution with no loss of efficiency in ocean models. The solution may lie in the use of the spectral element (SE) method. First developed in the early 1980s (Patera 1984), the spectral element method allows for irregular geometries and high accuracy because of polynomials of varying order inside quadrangles or triangles that form the mesh. We must stress that there is no intrinsic distinction between FE and SE methods except that the latter tend to be referred to as high-order FE schemes. In most SE methods, the basis functions are derived from Lagrangian interpolators at Gaussian-like points or from the Chebyshev or Legendre polynomials. As with the spectral method, the accuracy of the SE method increases exponentially with increasing polynomial order. However, the SE method offers more flexibility in geometrical representation. And, contrary to the spectral method where Gibbs oscillations are prone to occur in underresolved regions, one can easily increase the number of elements (h refinement) in the underresolved regions in the SE method. Using a polynomial order greater than two, we can also expect the SE method to be more accurate than conventional second-order FD or FE schemes with linear basis functions, and the convergence of the solution with increasing resolution to be much faster.

We note two applications in ocean modeling using a quadrangular SE method: Ma (1993), Iskandarani et al. (1995), and Iskandarani et al. (2003). Using quadrangles, it is relatively easy to construct an orthogonal basis of cardinal functions that renders trivial the matrix problem to be solved, provided the equations are prognostic and solved explicitly in time (leapfrog, Adams–Bashforth, Runge–Kutta). In this case, the modal coefficients associated with each basis function are also the actual physical values. Moreover, the associated collocation points are optimal for integrating polynomials over the quadrangle. The computation of nonlinear terms is therefore straightforward and does not require transfer from polynomial space to physical space. This makes the quadrangular SE method extremely efficient. One limitation of using continuous basis functions for the primitive (or shallow water) variables is that for stability, the maximum polynomial order for approximating pressure (or elevation) has to be lowered, compared to velocity (Iskandarani et al. 1995). On the contrary for the triangular SE approach, there is no orthogonal basis of cardinal functions in the triangle [although some orthogonal noncardinal basis exist (Lomtev and Karniadakis 1999; Taylor and Wingate 2000)]. Therefore, the computation of nonlinear terms requires a tedious transfer from spectral coefficients to values at physical Gaussian-like points, and then back to spectral space. However, automatic and local conforming mesh refinement is straightforward in triangular meshes. This is not as readily done with quadrilateral meshes. One exception is the use of nonconforming quadrilaterals (Mavriplis and Hsu 1997; Levin et al. 2000), which allows for direct subdivision of one quadrilateral without changes to the rest of the mesh.

In restricted applications, developments led to another family of SE methods. Cockburn et al. (1990) and later Lomtev and Karniadakis (1999, hereafter referred to as LK) used a discontinuous formulation that led to a local matrix problem in each element triangle. This is only possible if all the equations are prognostic (as they are for shallow water models) and treated explicitly in time. A hydrostatic Boussinesq ocean with a free surface can be modeled using this simplified spectral element method. Furthermore, LK’s model appears to be stable although the same set of basis functions is used for velocity and pressure. Additionally, the order of the basis functions can vary from element to element (in this study we use the same order everywhere), which allows for local $p$ refinement, that is, increasing locally the order of the polynomials. Finally, this method allows for an easy implementation of a time-variable adaptive mesh that we will introduce in section 4. The efficiency of this discontinuous SE method remains to be tested.

In this paper, we show comparisons between a second-order FD model based on the formulation of Sadourny (1975), a fourth-order FD model based on the numerics of the DIECAST model (Dietrich 1998), an FE model based on Lynch and Werner (1991), and a
discontinuous triangular SE model based on Lomtev and Karniadakis (1999). In section 2, we present our implementation of the LK model. Section 3 is devoted to linear and nonlinear tests and comparisons in fixed meshes and section 4 presents a simple automatic mesh refinement strategy and results. The conclusions are in section 5.

2. Discontinuous spectral element method

a. The basis functions

The matrix problem to be solved in each element is small for the order of the spectral element we choose to test (between three and seven). Therefore, the constraint of orthogonality over the set of polynomials for a cost-effective model is less stringent. We thus use an even simpler set of basis functions compared to LK, consisting of products of Legendre polynomials with a triangular truncation, with no use of degenerate points or collapsed local coordinates:

\[ \phi_l(\xi_1, \xi_2) = L_l(\xi_1)L_k(\xi_2), \quad l + k \leq n_e, \]

where \( n_e \) is the maximum order of the polynomials, \( \xi_1 \) and \( \xi_2 \) are the local coordinates inside the element (Fig. 1), and \( i \) is indexed as \( l \) runs from 0 to \( n_e \), and \( k \) runs from 0 to \( n_e - l \). The solution can be expressed inside the element \( j \) by

\[ f(\xi_1, \xi_2, t) = \sum_j a_j(t)\phi_j(\xi_1, \xi_2). \]

As an example, Fig. 1 shows \( \Phi_j = L_2(\xi_1)L_3(\xi_2) \). For the elements along the boundary, the projection of the basis functions \( (\Phi_j) \) onto another set of basis functions \( (\Phi'_j) \), which vanishes right at the boundary, ensures the use of different boundary conditions (no-normal flow, free slip, no slip, or inviscid):

\[ \phi'_j = L_i(\xi_1)[L_k(\xi_2) - (-1)^k]. \]

The projection consists of computing coefficients in the new basis using the relation

\[ (f', \phi') = (f, \phi'), \]

where the brackets represent a surface integral over one element. The new function \( f' \) can be expressed using the newly defined \( \phi' \) as \( f' = \Sigma_j a'_j\phi' \). Note that (4) corresponds to a least squares fit. Since the equations are expressed in terms of \( \phi' \), the coefficients \( a'_j \) have to be expressed in terms of \( a_j \). This is straightforward using (3). The different boundary conditions can also be implemented for elements sharing only one vertex with the wall. However, we were satisfied with the current implementation. Furthermore, in contrast to the continuous spectral element formulations and as in LK, we use the same polynomial order for all the variables. We did not encounter any stability problem related to this choice.

b. The discretized shallow water equations

As a first step, we apply the discontinuous spectral element method to the discretization of the shallow water equations. Using a weak formulation and the traditional notion of Galerkin methods, the system of equations inside each element triangle reads as follows:

\[ \begin{align*}
\frac{\partial u}{\partial t} \phi_i &= \left( \frac{\tau_x}{h}, \phi_i \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - f v, \phi_i \right) \\
&\quad + \left( g \eta_x \frac{\partial \phi_i}{\partial x} \right) - \iint g \eta_{\partial \phi_i} n_x \, ds \\
&\quad - \nu \left( \nabla u, \nabla \phi_i \right) - \nu \left( \nabla u_{\partial \phi_i}, n \phi_i \right) \\
\frac{\partial v}{\partial t} \phi_i &= \left( \frac{\tau_y}{h}, \phi_i \right) - \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + f u, \phi_i \right) \\
&\quad + \left( g \eta_y \frac{\partial \phi_i}{\partial y} \right) - \iint g \eta_{\partial \phi_i} n_y \, ds \\
&\quad - \nu \left( \nabla v, \nabla \phi_i \right) - \nu \left( \nabla v_{\partial \phi_i}, n \phi_i \right) \quad \text{and}
\end{align*} \]

\[ \begin{align*}
\frac{\partial \eta}{\partial t} \phi_i &= \langle \mathbf{v} (\eta + H), \nabla \phi_i \rangle \\
&\quad - \iint (\eta + H)_{\partial \phi_i}(\mathbf{v}_{\partial \phi_i} \cdot \mathbf{n}) \phi_i \, ds.
\end{align*} \]
The viscosity is applied through a Laplacian operator 

\[ \nabla^2 \text{v} \]

in the momentum equations. In a discontinuous spectral element model, the Laplacian operator of the velocity components in the diffusion terms cannot be computed directly (see LK for details). The computation has to be done in two steps. First, the gradient tensor of the velocity is computed using a weak formulation and an integration by parts. The mass matrix is then inverted:

\[ \left( \frac{\partial v}{\partial x} \phi \right) = -\left( u, \frac{\partial \phi}{\partial x} \right) + \int u_{\phi} \phi \, n_x \, ds. \tag{9} \]

Second, the divergence of the gradient terms in the Laplacian is computed in the momentum equations again using an integration by parts. The interelement boundary terms are again computed as the mean of both sides. This ensures a smaller discontinuity between elements for the gradient terms. The gradient terms are then applied in (5) and (6). The dynamical boundary condition used is free slip. This is implemented in the velocity gradient tensor by forcing the normal derivative of the tangential velocity to zero at the wall, using the special set of basis functions defined by (3).

The nonlinear terms are somewhat expensive to compute. They require a transformation of the local spectral coefficients to a local set of Gaussian points (the physical space) and a back transformation to the spectral space. The choice of the right Gaussian points is obviously important. After trying different options, we use irregular points on the triangle (Lyness and Jespersen 1975; Dunavant 1985), which are more optimal at low order than a tensor product of Gaussian points. These points allow exact transformations between spectral and physical spaces, but are unfortunately only given for \( n_v \leq 10 \). For higher degrees, it is always possible to use a tensor product of Gauss or Legendre–Lobatto points (Iskandarani et al. 1995).\(^4\) Finally, For \( n_v = 0 \), we note that the discontinuous SE formulation is equivalent to a nonstaggered FV method.

The time integration is done using a fourth-order Runge–Kutta method. This choice would give equivalent space and time truncation errors for polynomials of degree \( n_v = 5 \). The spectral element shallow water model is hereafter referred as SPOC (spectral ocean; see Dupont 2001).

3. Test cases for meshes fixed in time

We choose to compare the accuracy of the discontinuous spectral element model (SPOC) against two versions of a well-known finite-element code in coastal oceanography known as Quoddy (Lynch and Werner 1991; Lynch et al. 1996). From Quoddy, we derive two barotropic shallow water versions: finite element method (FEM) lumped and delumped. The “FEM lumped” uses the same lumping technique as in Quoddy, which saves computer time at the expense of some loss in accuracy. The “FEM delumped” version solves a matrix problem for the velocity components and the elevation. This version is more computationally expensive but is more accurate. The time integration in both versions is only second-order accurate. A third model that we use is a second-order-accurate finite-difference model (FD2), based on a potential enstrophy conserving C-grid model (Sadourny 1975). Finally, we use a fourth-order-accurate finite-difference model (FD4) based on the numerics of the DIECAST model (Dietrich 1998).

In this model, some numerical operations close to the boundary are only second-order accurate. This may influence the overall accuracy. In both FD models the time integration is done using a fourth-order Runge–Kutta integration scheme with no time filters, which are usually required with leapfrog schemes. The meshes for the finite element and spectral element are unstructured.

\(^4\) Interestingly, Taylor and Wingate (2000) proposed to use Fekete points along with an approximate integration. They claim that the savings in cost outweighs the loss in accuracy.
while the finite difference method (FDM) uses a structured grid. The basin geometry is a square domain. This choice favors the FD models, their discretization axes being parallel to the domain boundaries.

a. Conservation properties

The test case of an isolated geostrophic eddy is used to examine the conservation properties of the FE, FD, and SE models. The total energy should be conserved if no dissipation or friction is applied, as the initial geostrophically balanced eddy moves slowly westward due to the sphericity and rotation of the earth. The Coriolis and $\beta$ parameters associated with the rotation of the earth are $f_0 = 1.0285 \times 10^{-4} \text{ s}^{-1}$ and $\beta = 1.607 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$. The domain is 1000 km $\times$ 1000 km. The reduced gravity is set at $g' = 0.01 \text{ m s}^{-2}$ to slow down the gravity waves. The initial maximum elevation for the Gaussian eddy is 580 m and the $e$ radius is 100 km. The elevation and velocity fields are initially in an approximate geostrophic balance. The mesh used by the FE model has 11908 elements and 6065 nodes, whereas the mesh used by the SE model has 132 elements and 80 nodes. The grid for the FD model has 101 by 101 points. The results (Fig. 2) show a significant dissipation in the FE model. This is due to the special formulation of the elevation equation needed for the stability of the model. The lumped version gives poorer results than the delumped case. The SE model (mesh and elevation after 18 days are shown in Fig. 3) does not require any damping or special treatment in order to stabilize the model. Hence, the conservation properties are as good, if not better, than the finite-difference model.

b. Accuracy and convergence for the linear wave test

In this section, we present results for the linearized SW gravity wave propagation problem in a square domain. This test is extremely crude but sufficient to show the dependence of each model to the order of the truncation errors. An elevation perturbation is imposed at the beginning of the simulation, in the form of a cosine wave with phase lines parallel to the $y$ axis:

$$\eta(x, y, t = 0) = \eta_0 \cos(2\pi x/L_x).$$

The initial velocity is zero. The wave propagates along the $x$ axis. Since there is no dispersion in the $y$ direction, the problem simplifies to a one-dimensional problem and a simple analytical solution can be found. An equation in $\eta$ can be found by substituting the $u$ equation in to the $\eta$ equation:

$$\partial_t \eta - g' H \partial_x \eta = 0.$$  

With no-normal-flow boundary conditions, the solution is

$$\eta(x, y, t) = \eta_0 \frac{2\pi x}{L_x} \cos(o t).$$

where $\omega = \sqrt{g' H^2 \pi/L_x}$. Therefore, the wave is a free mode of oscillation for the square basin. It bounces back and forth between the walls at the period of $2\pi/\omega$. The velocities are given by

$$u(x, y, t) = u_0 \sin\left(\frac{2\pi x}{L_x}\right) \sin(\omega t),$$

$$v(x, y, t) = 0$$

where $u_0 = g' H_0/\sqrt{g H}$.

To make the results independent of the time integration technique, we choose the value of the Courant number to be rather low at 0.1, with the numerical simulation performed up to 1/10th of the characteristic period of the wave. The goal of this test is to emphasize the importance of the spatial truncation errors and the corresponding computational cost and that the short model run is sufficient for this purpose. By increasing the resolution of the models and comparing the numerical solution to the analytical solution, we can compute the errors and the effective truncation order of each scheme. The velocity errors are computed as follows. For the FD models, the grid is oriented along the walls of the
square that coincide with the direction of the wave propagation, the $x$ axis. Hence there are no dispersion errors in the $y$ direction. However, the FE and SE methods use irregular meshes consisting of triangles that are randomly oriented. Therefore, these methods show a dispersion error along the $y$ axis that can be quantified as a function of resolution. A velocity rms error is computed and normalized as

$$
E(v_{\text{mod}}) = \sqrt{\frac{\iint (v_{\text{mod}} - v_{\text{anal}})^2 \, dx \, dy}{\iint dx \, dy}},
$$

(14)

where $v_{\text{anal}}$ and $v_{\text{mod}}$ represent, respectively, the analytical and model velocities. Term II is computed analytically and is the same for all models. For the FD model, term I is approximated by $\sum_{ij} \|v_{\text{mod}} - v_{\text{anal}}\|^2/(n_x n_y)$. For the FE and SE models, term I is computed by interpolating $v_{\text{mod}} - v_{\text{anal}}$ onto a regular grid, summing the values, and dividing by the number of sampling points. We increase the number of sampling points until a convergence criterion is satisfied. The choice of norms in (14) in determining the error is somewhat arbitrary and other norms can be used. However the results would not be substantially different.

### Table 2. Convergence order for the different models for the linear wave experiment in a square domain and for the nonlinear Munk problem.

<table>
<thead>
<tr>
<th>Model</th>
<th>Linear test</th>
<th>Nonlinear test</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD2</td>
<td>1.98</td>
<td>2.18</td>
</tr>
<tr>
<td>FD4</td>
<td>3.24</td>
<td>—</td>
</tr>
<tr>
<td>FEM</td>
<td>1.80</td>
<td>—</td>
</tr>
<tr>
<td>SPOC 3</td>
<td>2.73</td>
<td>—</td>
</tr>
<tr>
<td>SPOC 5</td>
<td>4.66</td>
<td>4.96</td>
</tr>
<tr>
<td>SPOC 7</td>
<td>6.51</td>
<td>—</td>
</tr>
</tbody>
</table>

For the FE and SE models, the use of irregular grids would cause errors to appear in the $v$ component, perpendicular to the propagation direction. These errors can also be viewed as a dispersion error. One way to minimize this error would be to design meshes for which the nodes or vertices are aligned with the propagation axis (i.e., characteristic methods). Such a mesh would therefore be application dependent. We focus instead on the use of irregular meshes in which the triangles are randomly oriented since there are generally no preferred directions of propagation. For the FD model, the spatial resolution is defined as the inverse of the distance between two neighboring grid points. The FE resolution is taken to be the inverse of the mean length of the sides of the triangles. To make results comparable, the mean length in the SE mesh is divided by the maximum polynomial in order to yield an SE resolution. For the FE model, we show results using only the lumped version of the model. For this test case, both the lumped and delumped versions give similar results, with the delumped version being somewhat less cost effective.

Figure 4 shows the convergence of the error with resolution for a linearized version of the models. The errors are larger for FEM compared to FD2 and FD4 due to the use of unstructured grids in FE models. The slopes of the curves indicate the order of the truncation errors. It is about 2 for FD2 (see also Table 2) indicating the model is second order. The convergence order for FD4 is closer to 3 due to second-order-accurate numerics along the boundary. The associated errors are nonetheless smaller than those of FD2 at all resolution. The FE convergence order is less than but close to 2 for the velocity vector (second column in Table 2). Hence, the effective order of the FE model is close to that of FD2, although the errors are larger. This seems to be an indication that irregular meshes tend to increase the overall error of the models but do not significantly modify the convergence order.

For the spectral element model and $n_c = 3, 5, 7$, errors are smaller than in either the FD or FE model (Fig. 4). The SE model has a convergence order between $n_c$ and $n_c - 1$ (second column in Table 2). If the basis functions were continuous, the best achievable convergence order would be $n_c = 1$. The loss of more than one order is probably related to the use of unstructured meshes and the fact that the basis functions are discontinuous be-
the inverse of the error. For instance, for the FD model, the number of grid points in one direction (i.e., the spatial resolution as defined above), and the accuracy, the inverse of the error. The further a point is to the right on the graph, the less error for the same cost. Assuming that points for one model follow a straight line, the model whose curve lies farthest to the right (left) is the most (least) cost effective. There is of course the possibility that model performance depends on the range of the required accuracy due to the existence of crossover points between the different curves. We can derive the slope of these curves based on the knowledge of the truncation order and the cost as a function of the resolution, assuming the fields are well resolved. We introduce $n_x$, the number of grid points in one direction (i.e., the spatial resolution as defined above), and $A$, the accuracy, the inverse of the error. For instance, for the FD model, the number of degrees of freedom in one direction is $n_x$, and in total $n_x^2$ (first column in Table 3). As the count number of operation per degrees of freedom is low, that is, $O(1)$, the cost per time step is of order $n_x^2$ (second column in Table 3). The maximum time step in our FD models is proportional to the resolution ($\sim 1/n_x$; third column in Table 3). Therefore the total cost for a specific simulation is of order $n_x^2$ (fourth column in Table 3). We verified that the error for a second-order-accurate FD model is proportional to $1/n_x^2$, that is, $A \sim n_x^2$. Hence, the cost in terms of accuracy should be of the order of $A^{1/2}$. The same is valid for the FE model. For the FD4 model the only difference comes from the higher order of the error convergence, although there is a slight increase in the count number of operations per degrees of freedom that we did not take into account. In the SE model, we took into account its dependence on the maximum polynomial order. Despite the triangular truncation, the number of degrees of freedom per element is still of order $n_x^2$. As the most expensive operation in one element is the matrix/vector multiplication and the matrix is full with rank $n_x^2$, the operation count is $n_x^4$ in each element. This yields a $n_x^4/n_x^2$ cost per time step. The time step in the SE model is constrained by $n_x$ but has also an $n_x$ dependence that is inversely proportional to $n_x^2$ due to the higher resolution close to the vertices and the edges of each triangle (Taylor et al. 1997). Hence, the total cost is $n_x^2/n_x^2$ (fourth column in Table 3). The convergence rate for the error is taken from Table 2. Table 3 shows that at the highest order ($n_x = 7$) the SE model will eventually, for sufficiently high resolution, be more cost effective, despite a more stringent limitation on the time step. This is basically true of any high order method.

Figure 5 shows the variation of the CPU cost with respect to accuracy for FD2 and FD4, the FE model, and the discontinuous SE model. The CPU time was computed on a 1-GHz AMD Athlon PC. The curves are mostly straight lines and are generally consistent with our discussion. The variations in the FD curves may be related to the efficiency of the cache memory of the machine. The FE model is always less accurate for the same cost with the slope being equivalent to that of FD2. The same can be said of the SE model with $n_x = 3$ and the fourth-order FD model. As discussed earlier, the SE model with $n_x = 7$ gives the best efficiency. This is not, however, in itself a new result (see Sanderson 1998). It only demonstrates that the discontinuous SE

![Figure 5: Same as in Fig. 4 but for the normalized error as a function of CPU cost.](image)

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**Table 3.** Approximate number of degree of freedom, cost per time step, time step dependence, total cost for a given simulation, error convergence, and cost as a function of accuracy for the four models. Here $n_x$ is the resolution as defined in section 3b and $A$ is the accuracy, the inverse of the error.

<table>
<thead>
<tr>
<th>Model</th>
<th>Degree of freedom</th>
<th>Cost per step</th>
<th>Time step</th>
<th>Total cost</th>
<th>Error convergence</th>
<th>Cost/accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>FD2</td>
<td>$n_x^2$</td>
<td>$n_x^2$</td>
<td>$1/n_x$</td>
<td>$n_x^2$</td>
<td>$1/n_x^2$</td>
<td>$A^{1/2}$</td>
</tr>
<tr>
<td>FD4</td>
<td>$n_x^2$</td>
<td>$n_x^2$</td>
<td>$1/n_x$</td>
<td>$n_x^2$</td>
<td>$1/n_x^2$</td>
<td>$A^{1/2}$</td>
</tr>
<tr>
<td>FEM</td>
<td>$n_x^2$</td>
<td>$n_x^2$</td>
<td>$1/n_x$</td>
<td>$n_x^2$</td>
<td>$1/n_x^2$</td>
<td>$A^{1/2}$</td>
</tr>
<tr>
<td>SPOC</td>
<td>$n_x^2/n_x$</td>
<td>$n_x^2/n_x$</td>
<td>$1/(n_xn_x)$</td>
<td>$n_x^2/n_x$</td>
<td>$1/n_x^2$</td>
<td>$n_x^2A^{1/2}$</td>
</tr>
</tbody>
</table>

3 This is in contrast to quadrilateral elements where tensor products limit the growth in the operation count to $n_x$. Some tensor products are possible for the triangle but they are most efficient at higher value of $n_x$ since they use a less optimal tensor set of Gaussian points.
method behaves similarly to FD methods with respect to the order of the scheme. Crossover points between the FD models and the SE model are not present in this figure because they occur at too low a resolution. This will be addressed in the next section.

c. Single-gyre Munk problem

In the FE model using linear basis functions, the free-slip boundary condition arises as the “natural condition” in the treatment of the viscous terms. In the FD model a change of the formulation of the viscous terms close to the wall is needed to ensure that the normal derivative of the tangential velocity is set to zero. Due to the enstrophy conserving formulation, a boundary condition for the vorticity is also required at the wall. Under free-slip conditions and along straight walls, this boundary condition corresponds to a vanishing vorticity.

A y-varying sinusoidal wind stress in the x direction is applied with a maximum amplitude of $10^{-4}$ m$^2$ s$^{-2}$ and a wavelength of 2000 km. After a spinup period, a gyre forms in the basin. The lateral viscosity stabilizes the western boundary layer by damping the strong western return flow. We retain only the delumped version of the shallow water FE model, which gives slightly better results. We compare results from the SE, FE, and FD models for two values of the viscosity coefficient. The SE model is run with $n_c = 5$ on a 56-element mesh. The FE model uses a mesh of 11 808 elements and 6105 nodes. For the high viscosity case ($\nu = 2000$ m$^2$ s$^{-1}$; Fig. 6) the models perform similarly, with the FE model showing smaller undershoots. The SE and FD2 kinetic energy curves are indistinguishable. For the SE model, Fig. 7 gives the contours of the elevation at the end of the simulation. No discontinuities are visible, despite the fact that the solution is discontinuous by definition. A larger discrepancy is observable for a lower viscosity case ($\nu = 700$ m$^2$ s$^{-1}$; Fig. 8) between the FE model and the FD and SE models, due to the dissipative nature of the chosen FE model. At the end of the 6-yr simulation, there is a 5% difference between the kinetic energy for the FD and SE models. This is evidence that the SE model lacks resolution in certain parts of the domain, as some discontinuities are now visible in the elevation field (Fig. 9).

d. Accuracy and convergence for the nonlinear Munk problem

We now examine the accuracy and cost of the FD and SE models. The test case is the one used in section 3c. We have discarded the solutions obtained by the FE model because of its dissipative behavior. As an indicator of the accuracy, we use the basin-integrated kinetic energy instead of the rms velocity error of section 3b.

Fig. 6. Kinetic energy during spinup for the single-gyre Munk problem with $\nu = 2000$ m$^2$ s$^{-1}$ for the C-grid FD, the delumped FE, and SE models. The FD and SE curves are indistinguishable. For the SE model (SPOC), $n_c = 5$ and the mesh has 56 triangles.

Fig. 7. Elevation field for the SE model after 6 yr from spinup for the single-gyre Munk problem with $\nu = 2000$ m$^2$ s$^{-1}$, $n_c = 5$, and a 56-triangle mesh. In geostrophic balance, the isoelevation lines can be taken as a proxy for the streamlines.

Fig. 8. As in Fig. 6 but with $\nu = 700$ m$^2$ s$^{-1}$.
Since the solution of this test problem is nonlinear, a reference solution is obtained by running the spectral model for 6 yr from rest with \( n_c = 7 \) and a mesh of 132 nodes. The error is then defined as the difference between the value of the kinetic energy after a 6-yr run and that of the reference solution. The normalized error is computed by dividing the error by the value of the kinetic energy of the reference solution. Figures 10 and 11 show the convergence of the error with resolution and CPU cost, respectively. The fact that the finite-difference results give close to second-order accuracy (third column in Table 2) suggests the reference solution is an accurate approximation of the true solution. These two figures confirm in general the behavior inferred from the linear test case. The convergence with resolution and CPU time is faster with higher-order methods. However, the fact that there is a crossover point indicates that below a certain resolution (\( \Delta x > 10 \) km), the FD model is more accurate for the same cost. The existence of this crossover point in this nonlinear test case is due to the additional cost of computing nonlinear terms and the dissipative operators in the SE model (approximately 70% at \( n_c = 5 \)), whereas the same operations are rather inexpensive in FD models. At the crossover point the error in kinetic energy is less than 1%. Therefore, the SE model is more cost effective than the FD model in a range of resolutions for which the overall error is already less than 1%. Unfortunately, there is only one point for \( n_c = 7 \), the second point being used as the reference solution. The location of this point is higher than expected from Fig. 5, again indicating a rather large operation count relative to the error.

4. Adaptive mesh refinement

Given the two to three order of magnitude difference between the scale of eddies and the basin scale, today’s global ocean models would require a variable resolution in time and space in order to resolve synoptic eddies. To fulfill this constraint, not only do we need a variable spatial resolution (which the FE and SE models already offer), but we also need some flexibility of the mesh varying in time, since eddies and fronts are unsteady phenomena. By adaptive mesh refinement, we mean that the mesh is refined automatically in regions where estimated errors are the largest, as the simulation progresses. The difficulty is in computing an error estimator that determines where to put more resolution. For FE methods using linear basis functions, it is usually recommended to estimate the local second-order derivatives of the fields and to have more resolution where these derivatives are the largest (Zienkiewicz and Taylor 1991, p. 571). Because the solution is piecewise linear, it is difficult to estimate its second-order derivatives. This usually requires the reconstruction of the solution by a higher-order method (Zhu and Zienkiewicz 1990). For continuous SE methods, adaptive strategies require an estimate of the slope of the spectral coefficient variation with wavenumber. If there is too much energy in the high wavenumbers, the elements have to be refined.
The parameter that controls whether an element is to be refined is only needed to complete the connectivity (Fig. 12). The re-refinement is hierarchical. When a triangle is to be cut into four, as the cutting into two children triangles is to be later re-refined, its parent triangle will be re-refined, it is cut into four children triangles and if one of the two children elements for each field. Though very simple, this estimator has not yet been found in the literature. Once the error estimator has been defined, the re-refinement or derefinement has not yet been found in the literature. From the point of view of defining an error estimator, the discontinuous SE method is slightly more effective. Since the proposed SE formulation allows the solution to be discontinuous between elements, a straightforward estimator is to compute the maximum jump between elements for each field. Though very simple, this estimator is to compute the maximum jump between elements for each field. Though very simple, this estimator has not yet been found in the literature. Once the error estimator has been defined, the re-refinement or derefinement of the mesh is fairly conventional and can be found in many textbooks, for instance in Zienkiewicz and Taylor (1991, p. 574). We finally end up with four parameters that control the re-refinement in time (see Table 4). The re-refinement is hierarchical. When a triangle is to be refined, it is cut into four children triangles and if the neighboring triangles are not to be refined, they are cut into two “children triangles” in order to have a conformal connectivity. But if one of the two children triangles is to be later refined, its parent triangle will be cut into four, as the cutting into two children triangles is only needed to complete the connectivity (Fig. 12). The parameter that controls whether an element is to be refined is $\lambda_1$. When the jump for one field through an interelement boundary is larger than the maximum value of that field over the entire mesh times $\lambda_1$, the two elements on both sides of the interelement boundary are to be refined. All steps of mesh re-refinement are kept in memory, thus easing the backward derefinement process. The parameter that controls whether an element is to be dereftined is $\lambda_2$. When the jump for every field through an interelement boundary is smaller than the maximum value of that field over the entire mesh times $\lambda_2$, the two elements on both side of the interelement boundary can be dereftined. The Courant–Friedrichs–Lewy (CFL) condition is updated every time the mesh is modified. The model requires a certain adjustment time to smooth the jump between elements after each mesh re-refinement. Therefore, there is a minimum value for $n_{\text{check}}$ (Table 4) depending on the time step and the physical parameters. The level of errors is regularly checked during the simulation. The model refines the mesh resolution accordingly, interpolates the fields onto the new mesh, and then restarts with the new mesh and fields. In contrast to steady flows for which the solution is unique (if the initial guess is close enough), the transient simulations present the disadvantage that the solution accuracy might degrade because the errors are still present in the new fields, although the resolution has been improved. This justifies the use of $\lambda_3$, the relative jump value above which the errors have reached an unacceptable level. In this case, the model should not restart from the present time step but from a previously saved time step at which the level of errors was smaller. The interpolation procedure from one parent element to the child elements is as follows. The field value of the parent element are computed at the Gaussian points of the child elements and transform back to spectral coefficients in the child elements. If a parent element has been cut already into two child elements and has to be refined, the spectral coefficients in the two child elements are used to compute the field values on the refined elements. The question of accuracy of adaptive time-stepping solutions also arises from the issue of interpolating the variable fields, since the interpolation does not conserve mass or energy. The Munk problem of section 3b was used to validate the adaptive strategy. We use the refinement parameters $\lambda_1$ and $n_{\text{check}}$ given in Table 4 and we test the SE model for the Munk problem with $\nu = 700 \text{ m}^2 \text{ s}^{-1}$ for three values for $\lambda_1$ (0.03, 0.02, and 0.01) and, respectively, three values of $\lambda_3$ (0.15, 0.10, and 0.05), which controls the maximum discontinuity allowable between two el-

![Fig. 12. An illustration of the remeshing strategy. The triangle to be refined is in gray.](image)
elements and when to restart the simulation from a previous time step. In Fig. 13 we show the circulation patterns of (middle panels) and meshes (top panels) at the end of the 6-yr simulation. The time evolution of the number of elements for $\lambda_1 = 0.01$ (the smallest value used) shows that part of the refinement goes into following the Kelvin waves at the beginning of the simulation, which requires more resolution along the boundaries (Fig. 13, bottom panels). When the Kelvin adjustment process becomes less important, a derefinement process occurs along the eastern and southern boundaries leaving higher-resolution regions along the strong western return flow. As $\lambda_1$ decreases, the refined triangles get smaller and smaller, and the total number of elements at the end of the simulation increases slightly. The isolines of the elevation field are smoother than those of Fig. 9, for which a fixed and rather uniform 56-triangle mesh was used for the SE model.

We now consider the cost effectiveness of the adaptive SE models compared to FD2 and to fixed-in-time meshes using the SE model. Since the adaptive SE allows for variable resolution in space and time, it may prove more effective than having a fixed and rather uniform mesh in time. The errors and costs for the adaptive strategy are given in Figs. 14 and 15. The convergence rate of the error in kinetic energy with resolution is better than that of the SE model at $n_c = 5$. However, the accuracy-to-cost convergence is not as good with the crossover point of the FD model being at a higher-accuracy level. This may be due to the fact that the refinement needed to resolve the Kelvin waves along the boundaries at the beginning of the simulation results in smaller time steps. This failure points to a need for local time stepping, although it is not quite clear how to implement such a procedure without loss of accuracy. Of note is that the error in the kinetic energy decreases faster than $\lambda_1$. For instance, we gain about one order in the kinetic energy error by decreasing $\lambda_1$ by a factor of 3. If the SE model were truly of truncation order $n$, close to the element edges, the kinetic energy error
should have decreased by the same factor as $\lambda_1$. This tends to prove that the errors in the SE model are larger at the boundary between elements where the discontinuities occur. However, these errors do not seem to adversely affect the overall accuracy, possibly because these larger errors are localized at the element edges.

5. Discussion and conclusions

The challenge of using irregular grids (i.e., with a high level of flexibility in resolution) with no loss of accuracy in ocean models is examined in this study using a discontinuous spectral element method, first proposed by LK. We examined the accuracy to cost function for a finite-difference, a finite-element, and the proposed spectral element model in a square domain, using linear and nonlinear test applications. Even though the square domain is the most favorable to the finite-difference model, the spectral element model shows a better accuracy for the same cost once the desired accuracy and spectral order are high enough. Essentially, this is equivalent to say that the errors introduced by an unstructured grid are only compensated when the order of the model is higher than that of the FD model. However, due to the larger cost of the SE model per degree of freedom, the desired accuracy has to be large (less than 1%) for the SE model to be more cost effective than the FD model.

For the nonlinear application, the finite-element model shows excessive diffusion, and no convergence of the solution was achieved. For the linear application, a finite-element model is always less cost effective than a finite-difference model of equivalent order because of the use of unstructured grids, which essentially increase the level of errors in finite-element models. Even if our FE or FV model were nondissipative and if a convergence of the nonlinear solution with resolution was achieved, it is very likely that the conclusions drawn from the linear case would stand, that is, such a model would be less cost effective at all resolutions than the FD model. From this point of view, the SE model appears to be a significant improvement. Of course, in the presence of irregular boundaries, the situation may be reversed, and a finite-element model may give better results than the finite-difference model.

A simple adaptive strategy for the discontinuous spectral element model was proposed that gives encouraging results, although the test case for validating the adaptive strategy and quantifying its cost effectiveness was subjective. One can argue that in this test case the need for adaptivity is simplistic and that a more stringent case would involve a moving front. Such a case would certainly be more difficult as the region of high resolution needs to move as the front propagates. We believe that the present approach can deal with this problem thanks to the use of the $\lambda_1$ parameter, which controls when to restart the model from previously saved fields (i.e., when the front was still well resolved). However, the cost will likely be larger because the model continuously restarts from earlier fields. In many situations though, an adaptive strategy gives at least a simple way of checking the local accuracy of the solution and provides a refined mesh that can be used later.

Future developments will include the use of the adaptive strategy with very low eddy viscosity values for the Munk problem, which allow for the formation of eddies, and the extension of our two-dimensional formulation to three dimensions.

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REFERENCES


