NOTES AND CORRESPONDENCE

An Update on Semi-Lagrangian Cost Effectiveness

PETER BARTELLO

Department of Atmospheric and Oceanic Sciences, McGill University, Montréal, Québec, Canada

STEPHEN J. THOMAS

National Center for Atmospheric Research, Boulder, Colorado, USA

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Corresponding author address: Prof. P. Bartello, Department of Atmospheric & Oceanic Sciences, McGill University, 805 Sherbrooke ouest, Montréal, (Québec) Canada H2A 2K6
Email: bartello@zephyr.meteo.mcgill.ca

Abstract

Two issues relating to the use of multidimensional spatial interpolation in semi-Lagrangian advection schemes are addressed. First, in response to the suggestion that a previous comparison by the authors of Eulerian and semi-Lagrangian schemes was at different orders of accuracy and was therefore misleading, it is demonstrated that the schemes were of the same order. In any event, the conclusion that semi-Lagrangian schemes are not cost-effective for mesoscale flows or flows influenced by topography is only weakly dependent on the order of the numerical approximations when attention is restricted to typical schemes occurring in the literature. Second, it is demonstrated that the most popular implementation of semi-Lagrangian advection, employing spatial interpolation at cubic order to advect the field, is unacceptably dissipative when compared to turbulence studies of the enstrophy cascade process.

1. The purpose of this note

Since our paper on this subject (Bartello and Thomas 1996; hereafter BT96) was published, the debate over whether semi-Lagrangian advection schemes are efficient at integrating mesoscale models influenced by topography has evolved. The present note aims to address two elements that have arisen since that time. First, in response to comments made in Côté *et al.* (1998) we separate the discussion of the relative merits of Eulerian versus semi-Lagrangian advection from the issue of the formal order of accuracy of both types of schemes. It was our intent to make fair comparisons and we argue here that we in fact did so. BT96 demonstrated that the fundamental difference in cost between the two families of schemes is due to the presence or absence of multidimensional interpolations and the trajectory calculation. The most important point of BT96 is that the Lagrangian time step cannot be increased significantly beyond the Eulerian time step for mesoscale flow or flow over topography, implying the extra cost of the semi-Lagrangian scheme due to numerous interpolations is not justified. The degree to which this is so is clearly dependent on the order of the numerical approximations in the two frames. However, when attention is restricted to reasonable comparisons between typical first, second or third-order schemes appearing in the literature, semi-Lagrangian advection is not recommended for these types of flows.

The second point also concerns interpolation; it allows us to make a quantitative comparison of the dissipation inherent in semi-Lagrangian schemes with estimates from turbulence studies of what that dissipation should be. We argue that semi-Lagrangian schemes (like all upwind-biased schemes) with third-order interpolation are unacceptably dissipative. Increasing the order of the interpolation alleviates this problem but makes semi-Lagrangian schemes even more prohibitively expensive. These two points are addressed in the following two sections and are summarized in Section 4, where it is concluded that cubic interpolation is both expensive and detrimental to the numerical solution. Alleviating one of these problems only serves to exacerbate the other.

2. Order of accuracy issues

In response to our paper Côté *et al.* (1998) stated that "The crux of their theoretical argument is that because there is no time step advantage for semi-Lagrangian advection for such flows, it is more cost effective to use a lower-order Eulerian advection scheme rather than a higher-order, more accurate but more expensive, semi-Lagrangian one." Since our original arguments did not depend crucially on the order of either Eulerian or semi-Lagrangian schemes, we intend to clarify our position on this issue.

At the end of Section 2 and in Figure 1 of BT96 we presented numerical simulations of 1-D advection comparing a popular semi-Lagrangian scheme to an Eulerian second-order centred-difference scheme in both space and time. The semi-Lagrangian scheme parallels the one described in Côté *et al.* (1998), namely it applies linear interpolation to the velocity field in the calculation of trajectories, followed by cubic interpolation on the advected field. Although Côté *et al.* (1998) refer to this discussion as "the only practical evidence given to support their claim", we presented it only to make the case that comparisons should not be made at equal Δt and Δx, but at equal computational effort. This demonstration showed that when the (in our view) appropriate comparison is made, the results can appear to be very different. The issue of whether Δt can or cannot be increased sufficiently, arguing from accuracy considerations alone, forms the crux of our study and is reported on elsewhere in BT96.

We continue to assert that such a comparison is fair. McDonald (1987) showed that if n is the order of the interpolation applied to the velocity in order to calculate a grid point's trajectory and m is the corresponding interpolation order used in advecting the field, then the global order of accuracy of the scheme is $\min(n+1, m)$. In this case the semi-Lagrangian scheme, like the Eulerian scheme, is second order, as it is in the model described in Côté *et al.* (1998) and in other studies referenced in BT96.

It is important to investigate in more detail the nature of the error as a function of the order of the scheme (e.g. LeVeque 1990, Strikwerda 1989). It can be isolated by solving the advection equation assuming that there is no time truncation error. A finite difference approximation to the first derivative of f at an interior point x_i on a uniform grid is given by

$$\frac{\delta f_i}{\delta x} = \frac{1}{\Delta x} \sum_{j=-N}^{N} w_j f_{i+j}, \tag{1}$$

where the weights, w_j, are chosen to yield the desired accuracy. The advection equation, $f_t = -U f_x$, is then approximated as

$$\frac{\partial f_i}{\partial t} = -U \sum_{n=1}^{\infty} W_n \Delta x^{n-1} \frac{\partial^n f_i}{\partial x^n},$$

where $W_n = \sum j^n w_j / n!$ with $W_1 = 1$. If we take the Fourier transform of $f(x,t)$ to be $\hat{f}_k(t)$, then the solution is

$$\hat{f}_k(t) = \hat{f}_k(0)\ \mathrm{e}^{-ikUt}\ A\ P,$$

where the numerical amplification factor is

$$A = \exp\Big(-W_2 k^2 \Delta x + W_4 k^4 \Delta x^3 + \ldots\Big) Ut$$

and the numerical phase factor is

$$P = \exp\ i\ \Big(-W_3 k^3 \Delta x^2 + W_5 k^5 \Delta x^4 + \ldots\Big) Ut.$$

The nature of the error reflects the leading-order error term for a given scheme, indicated by the lowest non-zero W_n. Here we note that for even-order schemes the error appears in the Fourier space amplitude whereas for odd-order schemes it is in the phase. The fact that the phase error is a function of the wavenumber k yields numerical dispersion.

If there is no error in the trajectory calculation, McDonald (1984) noted that semi-Lagrangian advection can be cast as a finite difference scheme whose order is the order of the interpolation. In this case the analysis above carries over to semi-Lagrangian schemes in that cubic interpolation yields amplitude errors and quadratic or quartic interpolation yields dispersion errors. Recently Leslie and Dietachmayer (1997) have shown numerically that this is true of the nonlinear problem of a rising thermal as well.

Following this Côté *et al.* (1998) stated "thus the issue raised by Bartello and Thomas (1996) is not really one of semi-Lagrangian versus Eulerian advection, but rather ... is it more cost effective to use a lower-order scheme at higher resolution or a higher-order one at lower resolution?" Our response to this statement is that we are not confusing the two issues since our demonstration consisted of a comparison of Eulerian and semi-Lagrangian advection at the same order. The semi-Lagrangian scheme appears to have error characteristic of a third-order scheme only because the advecting wind was constant and therefore error free. However, the point behind this demonstration was merely the cost of such a scheme for a more general problem. In this our comparison was both accurate and fair.

If we do consider comparisons of Eulerian and semi-Lagrangian schemes at equal arbitrarily high order, we can be guided by the operation counts required to perform multidimensional linear, quadratic and cubic interpolations tabled in BT96. It was also discussed in BT96 how the cost of performing sums of the form (1) is significantly inferior to 3-D interpolation as the order increases, since the sums (1) are one-dimensional at any order, whereas interpolation requires an increasing number of evaluations of cross derivatives.

Finally, we would like to emphasize the obvious conclusion of the above analysis and refer the reader to the work of Leslie and Dietachmayer (1997). It has been routinely argued by proponents of the semi-Lagrangian method that amplitude error is not as serious as phase dispersion error. This is not true when the solution is chaotic. However, it is often argued (e.g. Williamson and Laprise 1998 and references therein) that the lack of dispersion error in semi-Lagrangian models at cubic interpolation is a major advantage to semi-Lagrangian advection. We see now that, in fact, it has nothing to do with the reference frame and can be achieved much more inexpensively by applying a third-order Eulerian scheme.

3. Dissipation from Interpolation

Following McDonald's (1984) calculation of the amplification factors for a particular implementation of 2-D semi-Lagrangian advection of plane waves using Lagrange interpolation, McCalpin (1988) presented a detailed study of the dissipation inherent in a variety of semi-Lagrangian schemes due to the interpolation of the advected fields. The context was somewhat idealized since the variability in the dissipation due to spatially and temporally varying velocity fields was not addressed. However, if these dissipation rates are excessive for uniform advection, it is safe to assume that the fully nonlinear problem will show the same behaviour. McCalpin (1988) noted good agreement between his analysis and the fully nonlinear simulations of Ritchie (1988). In this section McCalpin's (1988) results are compared with estimates of the appropriate eddy viscosities for atmospheric flow obtained by Leith (1971) and Bartello, Métais and Lesieur (1996).

The amplification factors obtained by McDonald (1984) and discussed by McCalpin (1988) are

$$|\lambda_1|^2 = 1 - 2\alpha(1-\alpha)c$$

$$|\lambda_2|^2 = 1 - \alpha^2(1-\alpha^2)c^2$$

$$|\lambda_3|^2 = 1 - \alpha(2-\alpha)(1-\alpha^2)c^2[3 + 2c\alpha(1-\alpha)]/9$$

$$|\lambda_4|^2 = 1 - \alpha^2(1-\alpha^2)(4-\alpha^2)c^3[4 + c(1-\alpha^2)]/36,$$

where $|\lambda_n|$ refers to the amplitude amplification over one time step due to 2-D Lagrange interpolation at order n, α is the integral part of the Courant number (or residual Courant number) and $c = 1 - \cos k\Delta x$. McAlpin (1988) also examined the error resulting from cubic spline interpolation in the long-wave limit and found it to be only slightly less dissipative. Since it is of the same order it will not be examined in detail. Using these relations it is simple to define an effective eddy dissipation, $\nu(k)$ such that

$$|f(t+\Delta t)| = |f(t)|\ \mathrm{e}^{-\nu_n(k)\Delta t},$$

implying that

$$\nu_n(k)\Delta t = -\ln|\lambda_n|.$$

Note that since interpolation is applied at an instant yielding a total dissipation measured by $|\lambda|$, the rate of dissipation, $\nu(k)$, depends on the time interval between successive interpolations, Δt. In the turbulence studies below $\nu(k)$ depends only on flow parameters and is obtained independently of Δt. Direct comparison will therefore require specifying a value for Δt.

The quantities $\nu_n(k)\Delta t$ are displayed in Figure 1 at $\alpha = 0.5$, where it is again noted that even-order schemes are less dissipative than odd-order schemes. In spite of this the error associated with quadratic interpolation has led to the typical use of cubic interpolation in semi-Lagrangian models (see references in BT96). Below we compare this to the dissipation due to geophysical turbulence. Note that the interpolation dissipation is a maximum at the residual Courant number, α of 0.5. However, we quote McCalpin (1988): "... the decay time scale as a function of the residual Courant number, α, is near its shortest (i.e., most dissipative) value over most of the range of α, so that the worst-case viscosity coefficients calculated in [his] section 3a may be typical values if a broad range of residual Courant numbers are present." We will assume this to be the case.

Leith (1971) used two-point closure theory to deduce the effective eddy viscosity for a hypothetical model truncated in the enstrophy cascade of two-dimensional turbulence. Over this range the energy spectrum is

$$E(k) \sim \eta^{2/3} k^{-3}, \tag{2}$$

where η is the down scale enstrophy flux (for details see BT96). He obtained

$$\nu_L(k) = \eta^{1/3} f(k/k_{max}),$$

where the non-dimensional function, f, is plotted in Figure 2. Note that it reaches a maximum at $k = k_{max}$ of about 0.8. Here η is the enstrophy transfer rate and $\eta^{1/3}$

can be associated with the synoptic-scale root mean square vorticity, ζ_{rms}. Bartello *et al.* (1996) used a high-resolution numerical simulation of non-hydrostatic turbulence to obtain separate eddy viscosities for rotational and wave modes. They proposed

$$\nu_{BML}(k) = \eta^{1/3}\left[-0.03 + 0.5\left(\frac{k}{k_{max}}\right)^6\right]\left(\frac{k}{k_{max}}\right)^2$$

as the best fit to their simulation data for the rotational modes when the energy spectrum is (2).

We proceed by comparing these dissipation curves at $k = k_{max}$, assuming that they are reasonably similar functions of k. From the data of Figure 1

$$\nu_3(k_{max})\Delta t = 3.76, \tag{3}$$

while from Figure 2, Leith's (1971) statistical theory yielded

$$\nu_L(k_{max})\Delta t \approx 0.8\zeta_{rms}\Delta t$$

and the Bartello *et al.* (1996) simulation gave

$$\nu_{BML}(k_{max})\Delta t = 0.47\zeta_{rms}\Delta t.$$

In these equations the quantity ζ_{rms} represents the inverse of the average recirculation time of a synoptic-scale eddy, typically assigned a value of three days. Any numerical model must accurately resolve motion on this time scale since it describes the meteorological signal. Thus, many discrete samples must be calculated over the synoptic-scale period, implying $\zeta_{rms}\Delta t \ll 1$. As described in BT96, ζ_{rms}^{-1} is the Lagrangian time scale at all length scales when the spectrum is steep as in (2). In practice semi-Lagrangian models must respect the Eulerian time scale in flow over mountains and $\zeta_{rms}\Delta t$ must be even smaller. For example, if $\Delta t = 1$ hour and $\zeta_{rms} = 10^{-5}$ s^{-1}, then

$$\nu_L(k_{max})\Delta t \approx 0.03 \qquad \nu_{BML}(k_{max})\Delta t = 0.02.$$

These are seen to be over two orders of magnitude weaker than the inherent semi-Lagrangian dissipation at cubic interpolation in (3). For a mesoscale model the time step is smaller and the excessive nature of the dissipation is therefore much worse.

In fairness we should also examine the dissipation inherent in Eulerian schemes. Since semi-Lagrangian advection at Courant numbers less than unity is equivalent to upwind-biased Eulerian advection (e.g. Leslie and Dietachmayer 1997), we can conclude that these schemes are also too dissipative. However, there exist Eulerian alternatives, which dissipate much less and do not require expensive multidimensional interpolations. There are also Eulerian schemes which use interpolation (to estimate fluxes in finite-volume schemes for example) but which have minimal dissipation (e.g. Leonard, Lock and MacVean 1996). Smolarkiewicz (1984) minimizes dissipation and includes cross-derivative terms in order to stabilize higher-order variants of the MPDATA scheme. Although these are more expensive than derivatives of the form (1), they do not require the interpolation of the velocity field to calculate departure points and are therefore both less expensive and less dissipative.

4. Summary

In BT96 we demonstrated that the fundamental difference in cost between Eulerian and semi-Lagrangian schemes is due to the multidimensional interpolations and the trajectory calculation necessary in the latter. Second-order Eulerian centered finite difference schemes suffer from dispersion error. Third-order schemes suffer from amplitude error. Semi-Lagrangian schemes have very similar interpolation errors (Leslie and Dietachmayer 1997). In the chaotic turbulent flow of the atmosphere and oceans, amplitude error quickly generates phase error and *vice versa*. Our point of view is that it is merely total error that counts. Here we note that the semi-Lagrangian amplitude error is on a par with that of the most dissipative class of Eulerian schemes.

Côté *et al.* (1998) suggested that the discussion of BT96 confused the issues of grid refinement and increasing a scheme's order on the one hand, with the intinsic difference between Eulerian and semi-Lagrangian schemes on the other. We hope to have made it clear that our comparison was at an equal order of accuracy (MacDonald 1987). Schemes at much higher order are not very common in realistic meteorological and oceanographic models. In any event, at higher order multidimensional semi-Lagrangian interpolation is even more expensive than the one-dimensional sums (1).

Semi-Lagrangian schemes using cubic Lagrange (and cubic spline) interpolation were shown in Section 3 to dissipate resolved model scales much more than geophysical turbulence in the continuous problem. It is argued that inherent numerical dissipation over two orders of magnitude stronger than that implied by the cascade rate from the resolved to the unresolved motions is unacceptable. Of course, certain Eulerian schemes are also too dissipative, but this dissipation is built in to the semi-Lagrangian method and can only be reduced by a very expensive change to higher-order interpolation.

Finally, Côté *et al.* (1998) state "In our view the onus is still upon the proponents of the hypothesis that semi-Lagrangian methods are not cost-effective at the mesoscale to demonstrate that this is indeed so by performing comparative integrations under realistic conditions." We find this point of view surprising. Historically the behaviour of potentially interesting numerical schemes has been analyzed by employing simplified linear equations with constant coefficients. Desirable behaviour in this setting is deemed a necessary but not a sufficient condition for desirable behaviour in a fully-nonlinear realistic model. However, we have demonstrated that semi-Lagrangian schemes are not cost-effective even for the simple linear advection problem. We fail to see how these deficiencies can be cured by the added complication of a full model. In the particular case of geostrophic flow over topography (Section 4 of BT96) we proved that accuracy requires a very small Courant number, C. In spite of the fact that semi-Lagrangian techniques circumvent the stability criterion, $C < O(1)$, the accuracy criterion for this problem is $C \ll O(1)$. Rather than using a numerical model to validate the analytical solution obtained there, it seems more prudent to use the analytical solution to validate a model.

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